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Quantum response to classical transitions

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The richness and complexity of two-dimensional parameter space of the kicked Harper model is exploited to demonstrate *quantum fingerprints of all classical transitions*. The quantum system appears to follow the corresponding classical system in *parameter but not in time* in the localized, critical as well as in the ballistic regimes. Therefore, the relationship between quantum and classical systems appears to be universal when measured by their response to parameter changes. Additionally, a rather intriguing feature of quantum eigenstates is a set of critical points sprinkled in the regime where the classical dynamics is diffusive. These are the boundary points of the ballistic (localized) patches in the localized (ballistic) regime that survive in the semiclassical limit.

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The subject of classical-quantum correspondence in nonintegrable Hamiltonian systems is an open frontier in nonlinear dynamics. The fact that the quantum system could exhibit localized, diffusive, or ballistic behavior in the classically chaotic regime is an open puzzle in the forefront of fundamental physics [1]. In spite of the absence of direct classical-quantum correspondence, studies revealing quantum fingerprints of classical behavior are important to understanding the Ehrenfest theorem for nonintegrable systems. This paper focuses on exploring the quantum response to classical transitions in ballistic as well as localized phases of the quantum system. The kicked Harper model [2,3] is particularly suited for this as its two-dimensional parameter space is landscaped by an intricately nested phases [3,4] and the quantum dynamics can be ballistic, diffusive, as well as localized in the classically diffusive regime. The central result of our studies is that although quantum and classical may not follow each other in *time*, they follow each other in *parameter* and hence all classical transitions are *felt* in the corresponding quantum model. This suggests a rigorous footing for a universal relationship between classical and quantum transport *at all times*. This will be a generalization of a previously proposed relation [5,6] between classical diffusion D and the quantum localization length,

$$\xi = D/(2\hbar^2). \tag{1}$$

It should be noted that in the absence of dynamical localization, this equation is inadequate to capture a quantum response to classical transitions. One of the key points of this paper is that crystal clear quantum fingerprints of classical transitions are seen in not only the localized regime but also in the ballistic and critical regimes. Our detailed studies as described here suggest that classical-quantum correspondence may be established at all times within the *linear response to the change of parameters*.

The kicked Harper model is described by the time-dependent Hamiltonian

$$H(t) = L\cos(p) + K\cos(q)\sum_{k=-\infty}^{\infty}\delta(t-k). \tag{2}$$

Here q,p is a canonically conjugate pair of variables on a cylinder. For small $K(L)$, the transport is restricted along $p(q)$ due to Kolmogorove-Arnold-Moser (KAM) tori (Fig. 1). Interestingly, the KAM regime emanates as fractal tongues in two-dimensional parameter space. The tongues exist all the way up to $L\rightarrow\infty$ and appear to be exact replicas of each other after a nontrivial scaling of the parameters. Each tongue is approximately confined to a 2π interval in the L parameter while the corresponding K interval decreases asymptotically as $\approx 1/\log(L)$. For small K, the KAM to diffusive boundary appears to be linear in parameters.

For the quantum system, we investigate both the finite time and the infinite time dynamics. The former is studied with plane wave initial conditions using fast Fourier transform with N (up to 2^{19}) Fourier basis of unperturbed eigenstates. The latter is studied with very high precision using the renormalization group (RG) approach [4]. We consider irrational $\hbar$ values with a golden tail: $\hbar = 2\pi/\{nh + [\sqrt(5) - 1)/2$, where $\hbar$ is varied by varying integer nh. Clear fingerprints of classical KAM tongues with associated parametric periodicity appear in the corresponding quantum system where the quantum tongues correspond to the parameter re-

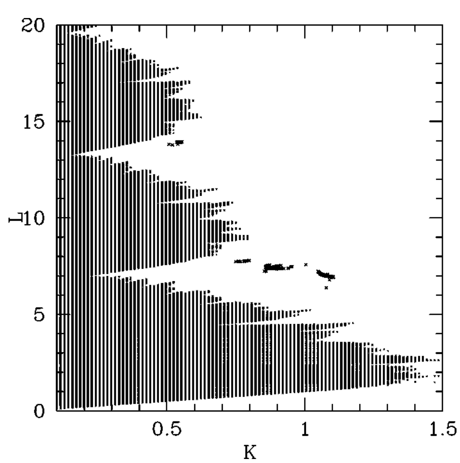

FIG. 1. The KAM to a diffusive boundary where the shaded region describes the KAM phase corresponding to bounded diffusion. The crosses show some of the parameter values with superdiffusive transport.

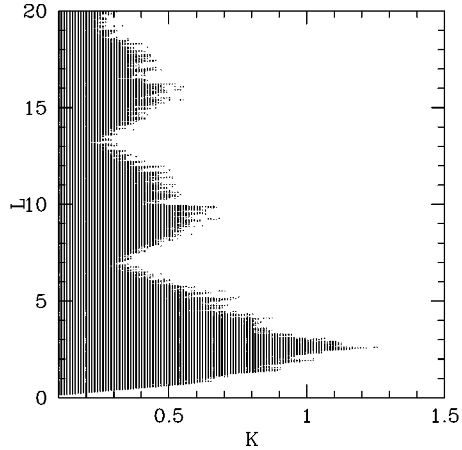

FIG. 2. Quasienergy phase diagram ($nh=100$) where the shaded part corresponds to the zero transmission probability beyond 100 angular momentum lattice sites. An almost identical figure is obtained using a quasienergy wave packet corresponding to the plane wave initial conditions.

gime with an extremely localized quasienergy state or a wave packet of quasienergy states. Our results demonstrate that the KAM to diffusive boundary in the quantum system is well described by Eq. (1). It should be emphasized that quantum manifestation of the cascade of KAM tongues as seen in Fig. 2 exists for small values of $\hbar$.

The diffusive regime in between the KAM tongues appears to be inhabited by the accelerator modes (AMs). These periodic orbits inducing anomalous transport have a very narrow stability window and hence require a very, very fine grid to see them. However, they are not missed by the quantum model and in fact the corresponding quantum peaks are very broad. Figure 3 shows a superdiffusive spike due to a period-15 AM [7] and a hierarchy of island chains deep inside the chaotic sea and the corresponding response of a localized quantum wave packet. In our detailed exploration of two-dimensional parameter space of the kicked Harper model, possibilities of the existence of AMs in the classical

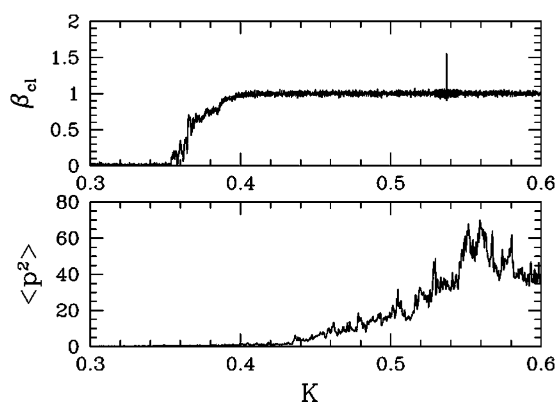

FIG. 3. Variation in classical (top) and quantum (bottom) transport ($L=13.9$) showing anomalous transport for very narrow parameter values in between the second and the third tongue of Fig. 1. The quantum results are for a quasienergy wave packet after 1000 time steps ($nh=200,N=2^{15}$).

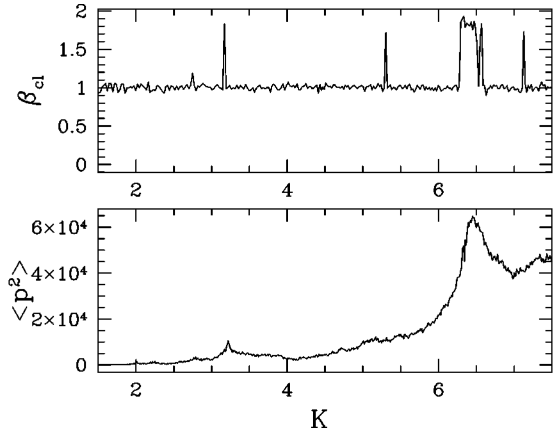

FIG. 4. Classical (top) and quantum (bottom) transport along the critical line $K=L$. The quantum results are for a quasienergy wave packet, after 1000 time steps with $nh=200,N=2^{15}$.

system were realized after spotting a rather broad quantum peak. This suggests that extremely tiny AMs existing in a very narrow parameter interval play an important role in the quantum transport. (Also see Figs. 4 and 5.)

The increase in the kinetic energy and hence the localization length of the quatum model whose roots are traced to the superdiffusive classical transport is more or less described by Eq. (1), provided D is interpreted as some measure of anomalous transport. Alternatively, this effect, which manifests itself as the enhancement of kinetic energy in both the quantum and the classical system, can be described in terms of the classical and the quantum obsevables such as $d\langle p^2\rangle/dK$. As we show below, this type of relationship between classical and quantum dynamics expressed in terms of $\langle p^2 d<p^2>/dK$ is found to persist in the ballistic as well as in the critical phase and hence in the regimes where Eq. (1) fails to describe the quantum response to the classical transitions.

Figures 6–8 show classical transitions and the corresponding quantum response in the absence of dynamical localization. The classical and the quantum transport continue

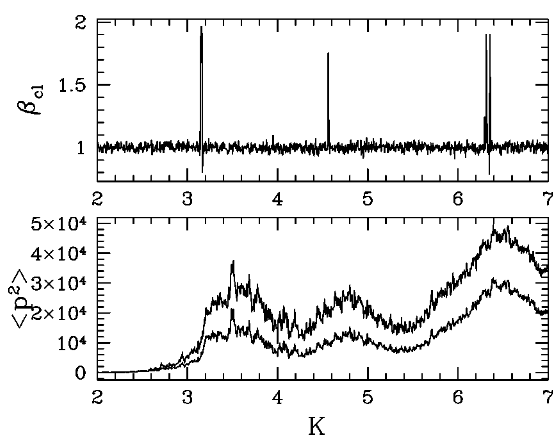

FIG. 5. Classical and quantum transport for $L=4$ showing the classical peaks due to superdiffusive transport and the quantum response to these transitions. Two curves describe the quasienergy wave packet after 2000 and 4000 time steps for $nh=200,N=2^{15}$.

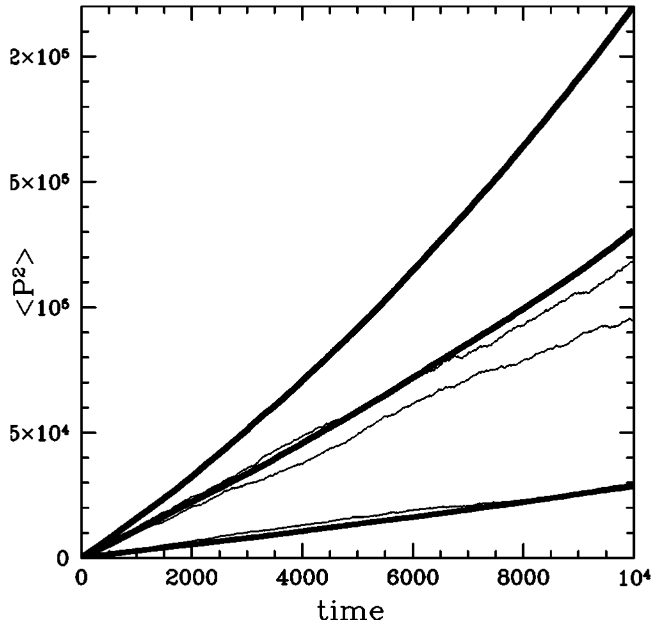

FIG. 6. Time evolution for some of the parameter values of Fig. 7 for fixed $L=4$ ($nh=200, N=2^{19}$). The thicker and thinner lines, respectively, show quantum and classical results, for (from top to bottom) $K=6.4, 6.2, 4$.

to follow each other in parameter along the symmetry line $K=L$ (Fig. 6) where the quantum transport is critical as well as in the ballistic phase ($K>L$) (Figs. 7 and 8). We would like to emphasize that quantum fingerprints of classical transitions in both these regimes cannot be described by relation (1) as these phases corresponding to infinite localization length. The figures clearly show enhancement of kinetic energy in the (critical and ballistic phases), demonstrating the awareness of the quantum system of the birth of AMs in the corresponding classical model.

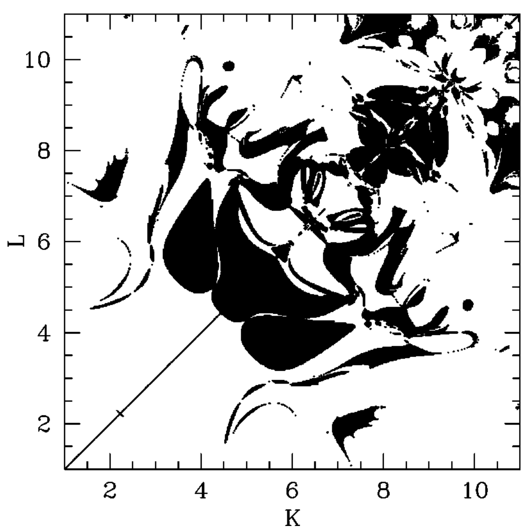

FIG. 7. Quantum phase diagram for the $\omega=0$ quasienergy state ($nh=10$). Shaded regions describe the localized (ballistic) phase for $K>L$ ($L>K$). Along the boundary of these shaded regimes reside the critical states.

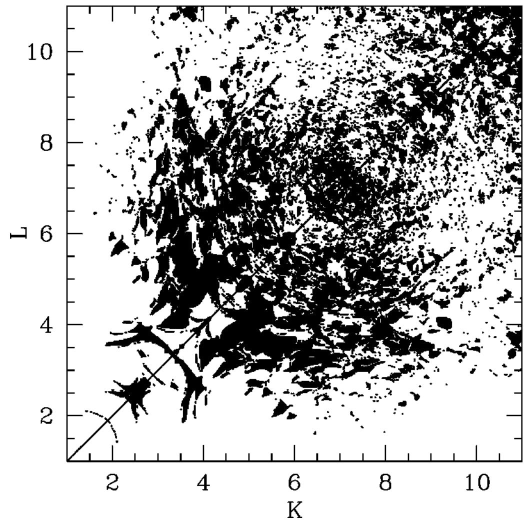

FIG. 8. Same as Fig. 4 for smaller $\hbar$. ($nh=30$) comparison of Figs. 3 and 4 suggest that as $\hbar\to 0$, the measure of the shaded regimes goes to zero and hence in the semiclassical limit, we have a set of parameter values (off the symmetry line) where the quantum states exhibit power-law localization.

Figure 8 compares time dynamics in classical and quantum systems in the regimes where the quatum dynamics is not localized. Interestingly, the classical and the quantum follow each other in both time and parameter only along the $K=L$ case. Therefore, *only* along the symmetry line $K=L$, where the quantum quasienergy states are critical [4], classical-quantum correspondence appears to be reasonably well described by the Eherenfest theorem. For $K>L$ the quantum transport is ballistic but the model continues to respond to classical phase transitions, such as the sudden triggering of superdiffusive behavior induced by the AMs just like the $L>K$ case when the quantum dynamics is localized. Therefore, the relationship between quantum and classical dynamics may be universal when measured by their response to parameter changes.

We now investigate localization characteristics of quasienergy states describing infinite time dynamics [4,3]. For large $\hbar$ values, we see patches of localized (extended) states for $K>L$ ($L>K$) where these two phases are intricately nested, as seen in Figs. 7 and 8. It appears that as $\hbar$ decreases, these finite regimes shrink in size. We conjecture that as $\hbar\to 0$, these localized (extended) regimes for $K>L$ ($L>K$) have zero measure and hence in the semiclassical limit all that remains are the boundary points where the quasienergy states are critical. Therefore, we have a scattered dust of points where the quantum states are critical, exhibiting power-law localization. Interestingly, the envelope of this scattered dust of critical points more or less coincides with the classically diffusive phase, a regime outside the KAM tongues of Fig. 1. Our detailed study of a quasienergy phase diagram for various values of $\hbar$ [9] suggest a new interpretation of quantum response to classically chaotic dynamics. The sea of critical points embedded in the ballistic (localized) regime provides a very appealing picture of the quantum fingerprints of classically chaotic dynamics [11], namely, the quantum manifestation of a classically diffusive phase is a regime which envelops a sea of critical points exhibiting diffusive transport.

Two key developments that are relevant to the question of quantum-classical correspondence are the concept of ‘‘break time’’ and a relationship between classical diffusion and the quantum localization length as described by Eq. (1). The work described here adds a new element to this as the quantum fingerprints of classical transitions exist in the localized, critical, as well as ballistic, phases. As a generalization of Eq. (1), we propose that the classical and the quantum derivatives of the observable with respect to parameters: for example, $d\langle p^2\rangle/dK$ are proportional to each other and hence may be a good candidate for establishing classical-quantum correspondence. In other words, we propose that the classical and the quantum behavior are related by a linear response theory.

The relation between quantum localization and classical dynamics in the quasi-integrable regime is pretty well understood mathematically by mean of a quantum version of the KAM theorem [8] involving control of the tunneling effect. The fact that the KAM phase that appears as a series of fractal tongues in two-dimensional parameter space is so well reproduced suggests that this argument is valid all the way up to the KAM diffusive boundary. The failure of the quantum system to follow the corresponding classical dynamics in the cantorus regime remains an open issue and a recent suggestion [3] that the self-similar cantorus potential may be the key should be further explored. However, the idea that the classical and the quantum systems appear to respond similarly to parameter changes is interesting and suggests that a relationship between the classical and the quantum behavior can be developed using an analytic approach within a linear response theory. We hope that this type of classical-quantum correspondence can be proven rigorously from first principles.

The kicked Harper model has attracted a great deal of attention for studying classical-quantum correspondence as well as for exploring superdiffusive transport in nonintegrable Hamiltonian systems [10]. A feature unveiled here is the series of KAM tongues which are believed to be replicas of each other under a scaling transformation. Preliminary studies suggest that the same scaling also applies to the kicked Harper mixed phase space: i.e., phase space structure at arbitrary K,L values can be related to some other values of $K'\leqslant K$ and $L<2\pi$. Additionally, $K=\pi/2$, which approximates the maximum K interval for the lowest KAM tongue appears to have a special significance. For $K>\pi/2$, parametric periodicity is simple: Phase space structure at K,L is related to the one at $K,L+2\pi n$. These intriguing details relevant to the kicked Harper model will be further investigated in the future.

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